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Compressible laminar boundary-layer flow at a point of attachment

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Howarth's transformation is applied to the compressible laminar boundary-layer equations for the flow at a point of attachment on a general curved surface. It is shown that the boundary-layer equations yield similarity solutions for the case when the viscosity varies linearly with the temperature, the Prandtl number is unity and the surface is maintained at constant temperature. The resulting eighth-order boundary-value problem is solved numerically for various surface temperature conditions and for various values of b/a, where a and b are constants representing the local velocity gradients in the principal directions of flow at the edge of the boundary layer.

Flow and heat transfer properties of the similar solutions at both nodal and saddle-points of attachment are given in graphical and tabular form.

1. Introduction

For the incompressible flow at a three-dimensional nodal point of attachment, Howarth (1951) has shown that the boundary-layer equations yield similarity solutions which are also exact solutions of the Navier–Stokes equations. The external flow is assumed to be irrotational and given by $\{ax_1, bx_2, -(a+b)x_3\}$, where x_1, x_2 are Cartesian co-ordinates of any point on the tangent plane at the stagnation point $x_1 = x_2 = 0$ and x_3 is measured along the normal at the stagnation point. Howarth discussed the properties of these solutions for $c = b/a = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and 1; the limiting values $c = 0$ and 1 corresponding to the two-dimensional and axially symmetric stagnation points, respectively. In a later paper Davey (1961) showed that similar solutions exist for $c \geqslant -1$. In Davey's terminology the similar solutions for $-1 \leqslant c < 0$ correspond to the flow near saddle-points of attachment and may in some cases be related to the flow in the vicinity of geometrical saddle-points on the surface. The main feature of Howarth's results is that rapid changes are produced in the boundary-layer flow as soon as the external stream departs from its two-dimensional form ($c = 0$); the results of Davey show that part of the boundary-layer flow is reversed when $c \leqslant -0{\cdot}4294$.

It is the purpose of this paper to discuss the corresponding compressible laminar boundary-layer flow at both nodal and saddle-points of attachment. The external flow is assumed to be irrotational and the surface is maintained at constant temperature. A transformation, due to Howarth (1948), is applied to the boundary-layer equations for a gas in which the Prandtl number is unity and the viscosity varies linearly with the temperature. The transformed boundary-layer equations possess similar solutions which are investigated numerically. These solutions are then used to discuss the dependence of skin friction and heat transfer on the external stream and surface temperature conditions. Note that the incompressible solutions obtained by Howarth (1951) and Davey (1961) correspond to compressible solutions in the present problem for the case of a thermally insulated surface. Other relevant solutions are the similar solutions of Cohen & Reshotko (1955) on two-dimensional and axially symmetric point flows for various surface temperature conditions.

2. The compressible laminar boundary-layer equations

Consider a system of co-ordinates x_1, x_2 orthogonal on the surface S and such that the element of length ds on S is given by

$$ds^2 = h_1^2\,dx_1^2 + h_2^2\,dx_2^2, \tag{2.1}$$

where h_1 and h_2 are functions of x_1 and x_2. If x_3 is measured along the normal to the surface at the point (x_1, x_2) the compressible laminar boundary-layer equations, as given by Stewartson (1964, p. 22), are as follows:

$$\frac{v_1}{h_1}\frac{\partial v_1}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial v_1}{\partial x_2} + v_3\frac{\partial v_1}{\partial x_3} + \frac{v_1 v_2}{h_1 h_2}\frac{\partial h_1}{\partial x_2} - \frac{v_2^2}{h_1 h_2}\frac{\partial h_2}{\partial x_1} = -\frac{1}{\rho h_1}\frac{\partial p_e}{\partial x_1} + \frac{1}{\rho}\frac{\partial}{\partial x_3}\left(\mu\frac{\partial v_1}{\partial x_3}\right), \tag{2.2}$$

$$\frac{v_1}{h_1}\frac{\partial v_2}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial v_2}{\partial x_2} + v_3\frac{\partial v_2}{\partial x_3} + \frac{v_1 v_2}{h_1 h_2}\frac{\partial h_2}{\partial x_1} - \frac{v_1^2}{h_1 h_2}\frac{\partial h_1}{\partial x_2} = -\frac{1}{\rho h_2}\frac{\partial p_e}{\partial x_2} + \frac{1}{\rho}\frac{\partial}{\partial x_3}\left(\mu\frac{\partial v_2}{\partial x_3}\right), \tag{2.3}$$

$$\rho c_p\left(\frac{v_1}{h_1}\frac{\partial T}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial T}{\partial x_2} + v_3\frac{\partial T}{\partial x_3}\right) - \left(\frac{v_1}{h_1}\frac{\partial p_e}{\partial x_1} + \frac{v_2}{h_2}\frac{\partial p_e}{\partial x_2}\right)$$
$$= \frac{\partial}{\partial x_3}\left(\frac{\mu c_p}{\sigma}\frac{\partial T}{\partial x_3}\right) + \mu\left(\left(\frac{\partial v_1}{\partial x_1}\right)^2 + \left(\frac{\partial v_2}{\partial x_2}\right)^2\right), \tag{2.4}$$

$$\rho T = \rho_e T_e, \tag{2.5}$$

and
$$\frac{1}{h_1 h_2}\frac{\partial}{\partial x_1}(\rho v_1 h_2) + \frac{1}{h_1 h_2}\frac{\partial}{\partial x_2}(\rho v_2 h_1) + \frac{\partial v_3}{\partial x_3} = 0. \tag{2.6}$$

Here (v_1, v_2, v_3) are the velocity components in the (x_1, x_2, x_3)-directions, respectively, T is the temperature and the pressure p_e is independent of x_3; the suffix e is used to denote the main stream condition. The dependence of viscosity on the temperature of the gas is taken as

$$\mu = (\mu_0/T_0)\,T, \tag{2.7}$$

where the suffix 0 is used to denote a reference condition in the main stream.

Let P be the stagnation point of attachment of the flow over S. It is now chosen as the origin of the co-ordinate system such that at P, $x_1 = x_2 = x_3 = 0$ and $h_1 = h_2 = 1$. If the surface is regular at P then on neglecting all terms $O(x_1, x_2)$ the relevant boundary-layer equations for the flow in the vicinity of a stagnation point are:

$$\rho\left(v_1\frac{\partial v_1}{\partial x_1} + v_2\frac{\partial v_1}{\partial x_2} + v_3\frac{\partial v_1}{\partial x_3}\right) = -\frac{\partial p_e}{\partial x_1} + \frac{\partial}{\partial x_3}\left(\mu\frac{\partial v_1}{\partial x_3}\right), \tag{2.8}$$

$$\rho\left(v_1\frac{\partial v_2}{\partial x_1}+v_2\frac{\partial v_2}{\partial x_2}+v_3\frac{\partial v_2}{\partial x_3}\right) = -\frac{\partial p_e}{\partial x_2}+\frac{\partial}{\partial x_3}\left(\mu\frac{\partial v_2}{\partial x_3}\right), \tag{2.9}$$

$$\rho c_p\left(v_1\frac{\partial T}{\partial x_1}+v_2\frac{\partial T}{\partial x_2}+v_3\frac{\partial T}{\partial x_3}\right)-\left(v_1\frac{\partial p_e}{\partial x_1}+v_2\frac{\partial p_e}{\partial x_2}\right)$$
$$= \frac{\partial}{\partial x_3}\left(\frac{\mu c_p}{\sigma}\frac{\partial T}{\partial x_3}\right)+\mu\left(\left(\frac{\partial v_1}{\partial x_1}\right)^2+\left(\frac{\partial v_2}{\partial x_2}\right)^2\right), \tag{2.10}$$

$$\rho T = \rho_e T_e, \tag{2.11}$$

and
$$\frac{\partial}{\partial x_1}(\rho v_1)+\frac{\partial}{\partial x_2}(\rho v_2)+\frac{\partial}{\partial x_3}(\rho v_3) = 0. \tag{2.12}$$

For Prandtl number $\sigma = \mu c_p/k = 1$ the energy equation (2.10) possesses a simple integral. On multiplying equation (2.8) by v_1, equation (2.9) by v_2, and adding these to equation (2.10), there follows the equation

$$v_1\frac{\partial H}{\partial x_1}+v_2\frac{\partial H}{\partial x_2}+v_3\frac{\partial H}{\partial x_3} = \frac{1}{\rho}\frac{\partial}{\partial x_3}\left(\mu\frac{\partial H}{\partial x_3}\right), \tag{2.13}$$

where
$$H = c_p T+\tfrac{1}{2}v_1^2+\tfrac{1}{2}v_2^2, \tag{2.14}$$

the approximate total enthalpy of the system. Thus, no matter what the viscosity law, if $\sigma = 1$,

$$H = H_e = H_0 = \text{const.} \tag{2.15}$$

is an integral of (2.10). It corresponds to a surface which is thermally insulated since

$$(\partial T/\partial x_3)_{x_3=0} = 0 \quad \text{when} \quad v_1 = v_2 = 0 \quad \text{(see Crocco 1946).}$$

Stewartson (1964, p. 62) has shown that even in the case when heat transfer occurs at the surface the energy equation (2.10) can be simplified. If the main stream is homenergic then

$$\tfrac{1}{2}(v_1^2+v_2^2)_e+\frac{a_e^2}{\gamma-1} = \tfrac{1}{2}(v_1^2+v_2^2)_0+\frac{a_0^2}{\gamma-1}, \tag{2.16}$$

where the suffix 0 is now used to denote a reference point in the main stream when $x_1 = x_2 = 0$. Stewartson introduces the dimensionless enthalpy function

$$S = (H-H_0)/H_0 \tag{2.17}$$

and the energy equation (2.13) is

$$v_1\frac{\partial S}{\partial x_1}+v_2\frac{\partial S}{\partial x_2}+v_3\frac{\partial S}{\partial x_3} = \frac{1}{\rho}\frac{\partial}{\partial x_3}\left(\mu\frac{\partial S}{\partial x_3}\right). \tag{2.18}$$

The temperature is now determined by the expression

$$\frac{T}{T_e} = 1+\frac{T_0}{T_e}\left(1+\frac{\gamma-1}{2a_0^2}(v_1^2+v_2^2)_0\right)S+\frac{\gamma-1}{2a_e^2}\{(v_1^2+v_2^2)_e-v_1^2-v_2^2\}; \tag{2.19}$$

in particular it may be shown that the surface temperature

$$T_w = (1+S_w)T_0, \tag{2.20}$$

where the suffix w denotes the condition at the surface.

The boundary conditions for the compressible laminar boundary-layer flow in the immediate neighbourhood of a point of attachment are:

$$\left.\begin{array}{ll}\text{at} & x_3 = 0, \quad v_1 = v_2 = v_3 = 0, \quad S = S_w, \\ \text{and} \quad \text{as} & x_3 \to \infty, \quad v_1 = ax_1, \quad v_2 = bx_2, \quad S \to 0.\end{array}\right\} \tag{2.21}$$

Following the method of Howarth (1948) it is now shown that the boundary-layer equations (2.8), (2.9), (2.11), (2.12), (2.18) and (2.19) can be partially reduced to the incompressible flow form. The Howarth transformation is

$$X_3 = \int_0^{x_3} \left(\frac{\nu_0}{\nu}\right)^{\frac{1}{2}} dx_3 = \left(\frac{p_e}{p_0}\right)^{\frac{1}{2}} \int_0^{x_3} \frac{T_0}{T} dx_3. \tag{2.22}$$

Stream functions (see Moore 1951) ψ and Φ satisfying the continuity equation (2.12) exist, and are defined as follows:

$$\rho v_1 = \rho_0 \frac{\partial \psi}{\partial x_3}, \quad \rho v_3 = -\rho_0 \left(\frac{\partial \psi}{\partial x_1} + \frac{\partial \Phi}{\partial x_2}\right), \quad \rho v_2 = \rho_0 \frac{\partial \Phi}{\partial x_3}. \tag{2.23}$$

These functions are modified by the transformations

$$\left.\begin{aligned}\psi(x_1, x_2, x_3) &= (p/p_0)^{\frac{1}{2}} \chi(x_1, x_2, X_3), \\ \Phi(x_1, x_2, x_3) &= (p/p_0)^{\frac{1}{2}} \phi(x_1, x_2, X_3);\end{aligned}\right\} \tag{2.24}$$

it follows that

$$v_1 = \partial\chi/\partial X_3, \quad v_2 = \partial\phi/\partial X_3, \tag{2.25), (2.26}$$

and

$$v_3 = -\frac{\rho_0}{2\rho p_0}\left(\frac{p_0}{p}\right)^{\frac{1}{2}}\left(\chi \frac{\partial p}{\partial x_1} + \phi \frac{\partial p}{\partial x_2}\right) - \left(\frac{p}{p_0}\right)^{\frac{1}{2}} \frac{\rho_0}{\rho}\left\{\frac{\partial \chi}{\partial x_1} + \frac{\partial \phi}{\partial x_2} + \left(\frac{\partial X_3}{\partial x_1}\right)\frac{\partial \chi}{\partial X_3} + \left(\frac{\partial X_3}{\partial x_2}\right)\frac{\partial \phi}{\partial X_3}\right\}. \tag{2.27}$$

The boundary-layer equations now become

$$\begin{aligned}\frac{\partial \chi}{\partial X_3}\frac{\partial^2 \chi}{\partial x_1 \partial X_3} + \frac{\partial \phi}{\partial X_3}\frac{\partial^2 \chi}{\partial x_2 \partial X_3} - \frac{\partial^2 \chi}{\partial X_3^2}\left(\frac{\partial \chi}{\partial x_1} + \frac{\partial \phi}{\partial x_2}\right) &= \left(v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2}\right)_e \left(\frac{T}{T_e} - \frac{\gamma}{2a_e^2}\chi \frac{\partial^2 \chi}{\partial X_3^2}\right) \\ &\quad - \left(v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2}\right)_e \frac{\gamma}{2a_e^2}\phi \frac{\partial^2 \chi}{\partial X_3^2} + \nu_0 \frac{\partial^3 \chi}{\partial X_3^3},\end{aligned} \tag{2.28}$$

$$\begin{aligned}\frac{\partial \chi}{\partial X_3}\frac{\partial^2 \phi}{\partial x_1 \partial X_3} + \frac{\partial \phi}{\partial X_3}\frac{\partial^2 \phi}{\partial x_2 \partial X_3} - \frac{\partial^2 \phi}{\partial X_3^2}\left(\frac{\partial \chi}{\partial x_1} + \frac{\partial \phi}{\partial x_2}\right) &= \left(v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2}\right)_e \left(\frac{T}{T_e} - \frac{\gamma}{2a_e^2}\phi \frac{\partial^2 \phi}{\partial X_3^2}\right) \\ &\quad - \left(v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2}\right)_e \frac{\gamma}{2a_e^2}\chi \frac{\partial^2 \phi}{\partial X_3^2} + \nu_0 \frac{\partial^3 \phi}{\partial X_3^3},\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}\frac{\partial \chi}{\partial X_3}\frac{\partial S}{\partial x_1} - \left(\frac{\partial \chi}{\partial x_1} + \frac{\partial \phi}{\partial x_2}\right)\frac{\partial S}{\partial X_3} + \frac{\partial \phi}{\partial X_3}\frac{\partial S}{\partial x_2} &= -\frac{\gamma}{2a_e^2}\left\{\chi\left(v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2}\right)_e + \phi\left(v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2}\right)\right\}\frac{\partial S}{\partial X_3} + \nu_0 \frac{\partial^2 S}{\partial X_3^2},\end{aligned} \tag{2.30}$$

respectively. Expression (2.19) completes the above system of equations. The appropriate boundary conditions are

$$\phi = \partial\phi/\partial X_3 = \chi = \partial\chi/\partial X_3 = 0, \quad S = S_w \quad \text{at} \quad X_3 = 0,$$

and
$$\partial\chi/\partial X_3 \to ax_1, \quad \partial\phi/\partial X_3 \to bx_2, \quad S \to 0 \quad \text{as} \quad X_3 \to \infty. \tag{2.31}$$

The flow is controlled by the parameters a, b and S_w. The constants a and b represent the local velocity gradients external to the boundary layer at the point of attachment. The dimensionless surface enthalpy function S_w determines the surface temperature by virtue of expression (2.20); the case $S_w = -1$ corresponds to a surface temperature at absolute zero, $S_w = 1$ corresponds to a surface temperature twice the external stream reference T_0, $S_w = 0$ corresponds to a surface temperature at the reference temperature T_0 and since $\sigma = 1$ this implies the case of a thermally insulated surface.

The transformed momentum equations (2.28) and (2.29) are nearly identical to the corresponding incompressible equations solved by Howarth (1951) and Davey (1961). The essential differences are the compressibility factors which multiply the pressure gradient terms.

3. The similarity solutions

The similarity solution of the transformed compressible laminar boundary-layer equations (2.28) to (2.30) subject to the boundary conditions (2.31) is expressed in the form

$$\chi = (\nu_0 a)^{\frac{1}{2}} x_1 f(\eta), \quad \phi = b(\nu_0/a)^{\frac{1}{2}} x_2 g(\eta), \quad S = S_w h(\eta), \tag{3.1}$$

where $\eta = a^{\frac{1}{2}} X_3/\nu_0^{\frac{1}{2}}$ is the dimensionless distance from the surface. The functions f, g and h satisfy the equations:

$$f''' + (f+cg)f'' + (1+S_w h - f'^2) = 0, \tag{3.2}$$

$$g''' + (f+cg)g'' + c(1+S_w h - g'^2) = 0, \tag{3.3}$$

$$h'' + (f+cg)h' = 0, \tag{3.4}$$

where $c = b/a$.

The boundary conditions for (3.2) to (3.4) are

$$\left.\begin{aligned} f = g = f' = g' = 0, h = 1 \quad &\text{when} \quad \eta = 0, \\ f' \to 1, \quad g' \to 1, \quad h \to 0 \quad &\text{as} \quad \eta \to \infty. \end{aligned}\right\} \tag{3.5}$$

This system of equations constitute an eighth-order boundary-value problem. At this stage reference may be made to certain solutions which have previously been obtained:

(i) $S_w = 0$ corresponds to the case of a thermally insulated surface. The basic equations for f and g have been solved by Howarth (1951) for $c = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and 1 and by Davey (1961) for values of c at intervals of 0·1 between 0 and -1. Since $S_w = 0$, corresponding solutions of the h-equation are not relevant to the present problem.†

† They do in fact relate to a physical problem in heat transfer. It is the problem of heat transfer at a point of attachment on a surface maintained at temperature T_1 when an incompressible constant property fluid ($\sigma = 1$) ahead of the surface is at temperature T_0. The local temperature distribution in the boundary layer is given by $T = T_0 + (T_1 - T_0)h(\eta)$.

(ii) $c = 0$ and 1 correspond to the compressible flow at two-dimensional and axially symmetric stagnation points, respectively. Cohen & Reshotko (1955) give solutions for $c = 1$, $S_w = 1, 0, -0{\cdot}4, -0{\cdot}8$ and -1; $c = 0$, $S_w = 1$. Moreover, sufficient information is given for estimates of the initial values $f''(0)$ and $h'(0)$ to be obtained by interpolation for $c = 0$, $S_w = -0{\cdot}4, -0{\cdot}8$ and -1.

S_w	c	$f''(0)$	$g''(0)$	$h'(0)$	$\left(\frac{a}{\nu_0}\right)^{\frac{1}{2}} \delta_1^*$	$\left(\frac{a}{\nu_0}\right)^{\frac{1}{2}} \delta_2^*$	$\frac{1}{S_w}\left(\frac{a}{\nu_0}\right)^{\frac{1}{2}} \delta_T^*$
1·0	1·0	1·7463	1·7463	−0·8102	0·441	0·441	0·723
	0·75	1·7383	1·5252	−0·7590	0·444	0·503	0·771
	0·50	1·7338	1·2736	−0·7076	0·442	0·591	0·828
	0·25	1·7332	0·9778	−0·6579	0·436	0·728	0·893
	0·0	1·7367	0·6156	−0·6156	0·425	0·959	0·959
	−0·25	1·7421	0·1623	−0·5935	0·413	1·361	1·005
	−0·50	1·7453	−0·3494	−0·6075	0·411	1·936	0·992
0·0	1·0	1·3119	1·3119	−0·7622	0·569	0·569	0·762
	0·75	1·2886	1·1643	−0·7139	0·588	0·629	0·814
	0·50	1·2669	0·9981	−0·6644	0·609	0·711	0·875
	0·25	1·2476	0·8051	−0·6153	0·629	0·832	0·947
	0·0	1·2326	0·5705	−0·5705	0·648	1·026	1·026
	−0·25	1·2251	0·2680	−0·5410	0·659	1·375	1·092
	−0·50	1·2302	−0·1115	−0·5484	0·655	1·962	1·089
−0·4	1·0	1·1246	1·1246	−0·7394	0·631	0·631	0·783
	0·75	1·0947	1·0084	−0·6923	0·659	0·691	0·836
	0·50	1·0652	0·8785	−0·6438	0·691	0·771	0·900
	0·25	1·0371	0·7290	−0·5947	0·725	0·884	0·976
	0·0	1·0121	0·5482	−0·5482	0·761	1·063	1·063
−0·8	1·0	0·9263	0·9263	−0·7134	0·703$_5$	0·703$_5$	0·807
	0·75	0·8894	0·8431	−0·6678	0·741$_5$	0·763	0·863
	0·50	0·8515	0·7512	−0·6202	0·786	0·840	0·929
	0·23	0·8130	0·6467	−0·5709	0·838	0·947	1·011
	0·0	0·7755	0·5219	−0·5219	0·898	1·109	1·109
−1·0	1·0	0·8219	0·8219	−0·6989	0·745	0·745	0·822
	0·75	0·7814	0·7560	−0·6541	0·788	0·804	0·878
	0·50	0·7389	0·6838	−0·6069	0·840	0·880	0·947
	0·25	0·6944	0·6026	−0·5574	0·903	0·984	1·032
	0·0	0·6489	0·5067	−0·5067	0·979	1·137	1·137
	−0·25	0·6081	0·3785	−0·4611	1·062	1·412	1·255
	−0·50	0·6030	0·1583	−0·4546	1·081	2·049	1·286

TABLE 1

Including the above mentioned known solutions the present paper gives solutions for the following cases:

$$S_w = 1, 0, -1 \quad \text{with} \quad c = 1(-0{\cdot}25)-0{\cdot}50;$$

$$S_w = -0{\cdot}4, -0{\cdot}8 \quad \text{with} \quad c = 1(-0{\cdot}25)0{\cdot}0.$$

These solutions of the boundary value problem (3.2) to (3.5) have been obtained using the Bristol University I.B.M. Computer. The integrations were performed using Gill's modification of the Runge–Kutta method. An iterative scheme was used for the determination of the unknown initial values $f''(0)$, $g''(0)$ and $h'(0)$.

Even with the use of a computer there are still considerable difficulties associated with the integration of the above equations especially for $c < 0$, as already discussed by Davey (1961) for the case $S_w = 0$.

An accuracy of six to seven places of decimals was obtained in the determination of the initial values $f''(0)$, $g''(0)$ and $h'(0)$. These have been rounded off to four decimal places and are given in table 1; they are also given graphically in figure 1 and values obtained by previous workers have been indicated.

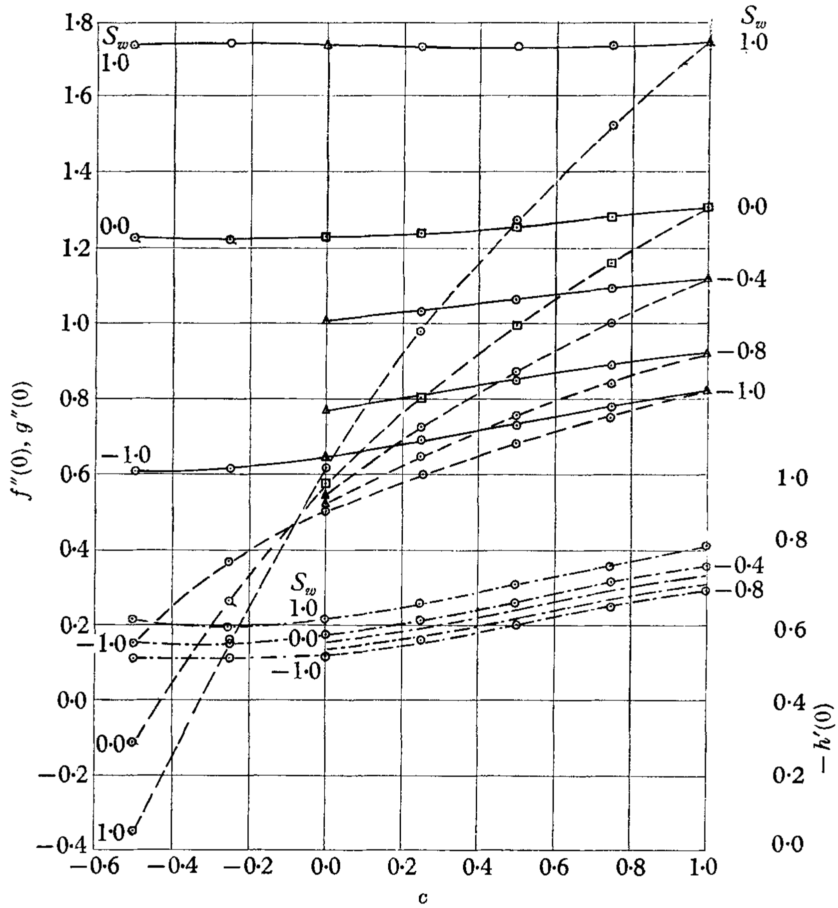

FIGURE 1. The initial values —— $f''(0)$, - - - $g''(0)$, – · – $h'(0)$. Howarth, ⊡; Cohen & Reshotko, △; Davey, ⊙; present report, ⊙.

In the transformed compressible plane the boundary-layer thicknesses δ_1^*, δ_2^* and δ_T^* are defined as follows:

$$\delta_1^* = \int_0^\infty \left(1-\frac{v_1}{v_{1e}}\right) dX_3 = \left(\frac{\nu_0}{a}\right)^{\frac{1}{2}} \int_0^\infty (1-f')\,d\eta, \tag{3.6}$$

$$\delta_2^* = \int_0^\infty \left(1-\frac{v_2}{v_{2e}}\right) dX_3 = \left(\frac{\nu_0}{a}\right)^{\frac{1}{2}} \int_0^\infty (1-g')\,d\eta, \tag{3.7}$$

and

$$\delta_T^* = \int_0^\infty S\,dX_3 = S_w\left(\frac{\nu_0}{a}\right)^{\frac{1}{2}} \int_0^\infty h\,d\eta. \tag{3.8}$$

These are given correct to three decimal places in table 1; δ_1^* and δ_2^* are given graphically in figure 2.

Tables of the velocity and thermal profiles are not given but some specimen profiles for $c = \pm 0{\cdot}5$ are given in figures 3 and 4†.

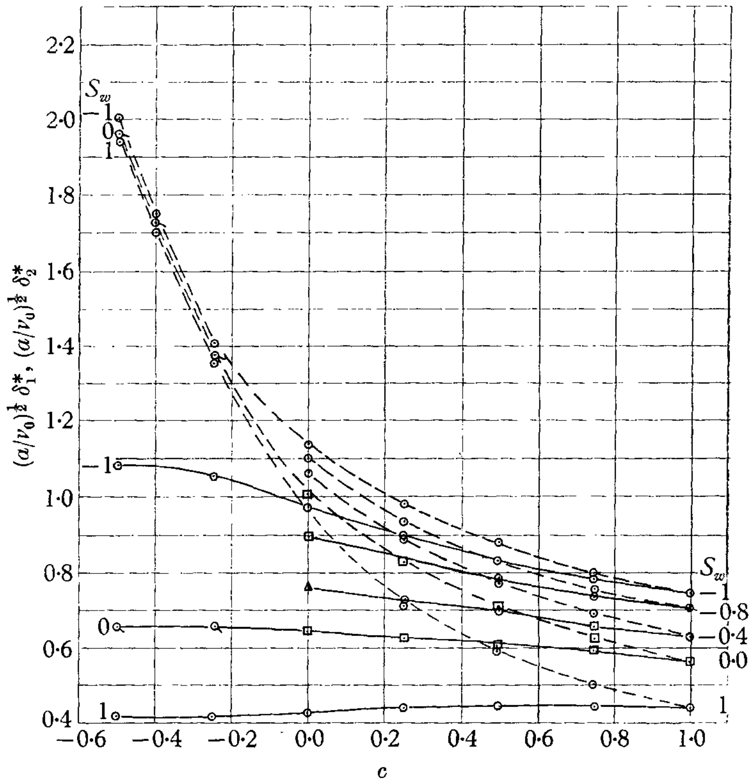

FIGURE 2. The boundary-layer thicknesses. ——, $(a/\nu_0)^{\frac{1}{2}}\delta_1^*$; – – –, $(a/\nu_0)^{\frac{1}{2}}\delta_2^*$. Howarth, ⊡; Cohen & Reshotko, △, Davey, ⊙̸; present report, ⊙.

4. Physical discussion of the results

In the following sections the similar solutions obtained in this paper are discussed. The two parameters defining a particular case are S_w and $c = b/a$. Briefly $S_w > 0$ or $S_w < 0$ defines a heated or cooled surface, respectively; for the flow in the vicinity of a nodal point of attachment c takes values in the range $0 \leqslant c \leqslant 1$, whilst for a saddle-point of attachment c takes values in the range $-1 \leqslant c < 0$.

Velocity and thermal profiles

For the case of a thermally insulated boundary it has already been established by Howarth (1951) that changes with c for nodal points of attachment are seen to be relatively small for the velocity component in the x_1-direction, but quite

† A table of values of f', g' and h for the twenty-four solutions obtained together with a table of h for the cases $S_w = 0$, $c = 1(-\frac{1}{4}) - \frac{1}{2}$ may be consulted by readers on application to the author or the editor.

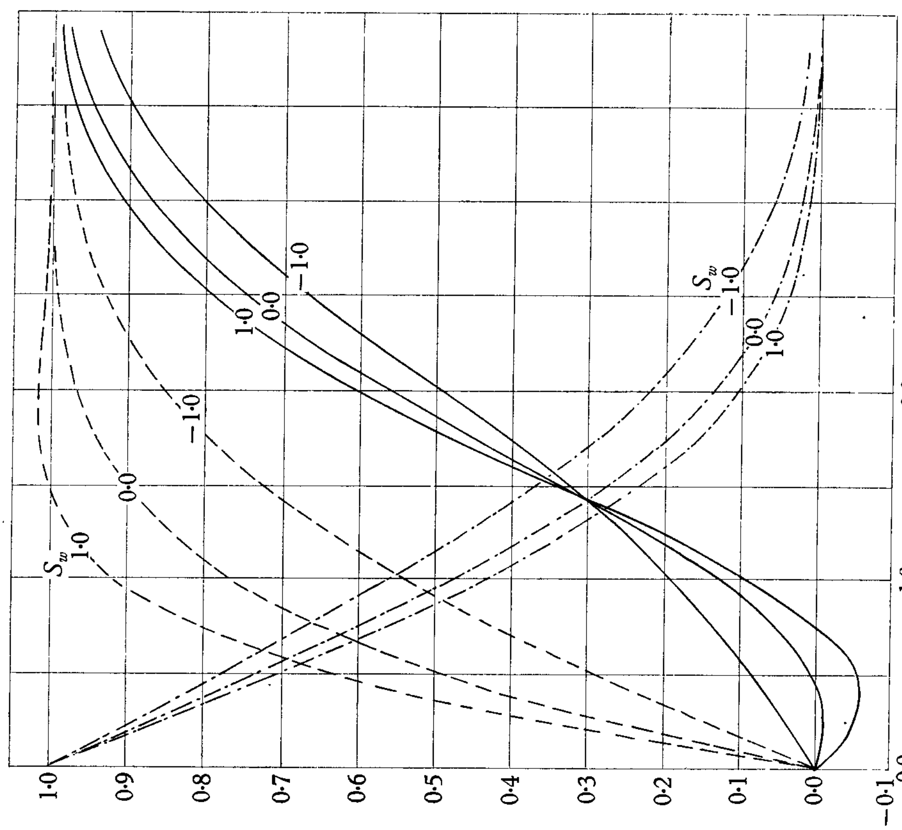

FIGURE 4. The velocity and enthalpy functions for $c = -0.5$.
f', - - -; g', ——; h, - · -.

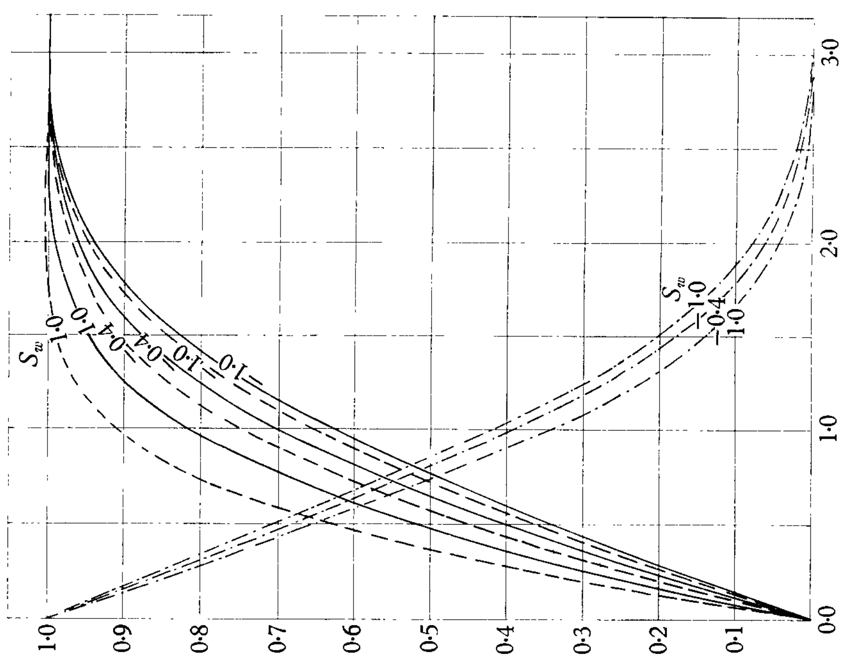

FIGURE 3. The velocity and enthalpy functions for $c = 0.5$.
f', - - -; g', ——; h, - · -.

marked in the x_2-direction. For saddle-points of attachment Davey (1961) has shown that part of the boundary-layer flow in the x_2-direction is reversed for $c < -0{\cdot}4294$.

Consider now the effect of heating or cooling the wall. Representative profiles for $c = 0{\cdot}5$, $S_w = 1$, $-0{\cdot}4$ and -1, and $c = -0{\cdot}5$, $S_w = 1$, 0 and -1 are given in figures 3 and 4, respectively. For a nodal point of attachment on a heated wall given by the parameters $c = 0{\cdot}5$, $S_w = 1$ there is a slight velocity overshoot of order 0·3 % of the terminal velocity in the x_1-direction. This is due to the fact that when heat is supplied to the boundary layer the density is reduced, with consequential increase in flow near the edge of the boundary layer. When the surface is cooled no velocity overshoot is seen to occur in the x_1-component of velocity.

c	1·0		0·75		0·5		0·25		0·0	
S_w	ϵ	cx_2/x_1	ϵ	cx_2/x_1	ϵ	cx_2/x_1	ϵ	cx_2/x_1	ϵ	cx_2/x_1
1·0	0°	1·00	4°	1·07	9°	1·17	16°	1·33	28°	1·68
0·0	0°	1·00	3°	1·05	7°	1·13	12°	1·25	21°	1·47
−0·4	0°	1·00	2°	1·04	5°	1·10	10°	1·19	17°	1·36
−0·8	0°	1·00	1°	1·03	4°	1·06	6°	1·12	11°	1·21
−1·0	0°	1·00	1°	1·02	2°	1·04	4°	1·07	7°	1·13

TABLE 2

However, for a saddle-point of attachment on a heated wall defined by the parameters $c = 0{\cdot}5$, $S_w = 1$ the velocity overshoot in the x_1-direction is of order 2 % of the terminal value. Moreover, the extent of flow reversal in the x_2-direction is considerably increased as compared with the insulated case for $c = -0{\cdot}5$. Once again the effect of cooling the wall prevents velocity overshoot in the x_1-direction and can even delay the occurrence of flow reversal in the x_2-direction.

It is interesting to consider the changes in direction of the velocity vector in passing through the boundary layer. There is a limiting direction of flow on the surface which is also the direction of resultant skin friction. This direction is inclined to the main stream at an angle ϵ, where

$$\epsilon = \tan^{-1}\left(\frac{cx_2\, g''(0)}{x_1\, f''(0)}\right) - \tan^{-1}\left(\frac{cx_2}{x_1}\right). \tag{4.1}$$

The maximum values of changes in direction and the corresponding values of (cx_2/x_1) are given in table 2 for nodal points of attachment. The values when $c = 0$ are limiting values as $c \to 0$, and should not be confused with the two-dimensional situation in which $c = 0$ and $\epsilon = 0$; but for any departure from the two dimensional case the results of table 2 will apply. It is seen from this table that the most marked effect occurs when the surface is heated.

Figures 3 and 4 show that the enthalpy function, and hence the temperature, varies monotonically across the boundary layer from surface to the free-stream value. The effect of surface temperature on the temperature variation across the boundary layer is more marked in the case of a saddle-point of attachment.

Boundary-layer thickness

From table 1 and figure 2 it is seen that the boundary-layer thicknesses δ_1^*, δ_2^* and δ_T^* increase with decreasing surface temperature. Note that δ_1^* and δ_2^* tend to reach their maximum values near when flow reversal just occurs, and then appear to decrease once it has been established.

Shear stress

The shear stress components at the surface are given by

$$\tau_{1w} = \rho_w \nu_w^{\frac{1}{2}} a^{\frac{3}{2}} x_1 f''(0) = \rho_0 \nu_0^{\frac{1}{2}} a^{\frac{3}{2}} (T_e/T_0)^{\gamma/\gamma-1} x_1 f''(0) \tag{4.2}$$

and

$$\tau_{2w} = c\rho_w \nu_w^{\frac{1}{2}} a^{\frac{3}{2}} x_2 g''(0) = c\rho_0 \nu_0^{\frac{1}{2}} a^{\frac{3}{2}} (T_e/T_0)^{\gamma/\gamma-1} x_2 g''(0). \tag{4.3}$$

The quantities $f''(0)$ and $g''(0)$ are given in table 1 and figure 1. It can be seen that heating the wall increases the sensitivity of the wall shear to pressure gradient especially in the x_2-direction. For $S_w = 1$ the quantity $f''(0)$ reaches a minimum value near $c = 0{\cdot}25$ and then increases; flow reversal occurs in this case when $c = -0{\cdot}33$. The corresponding figures for minimum value of $f''(0)$ and vanishing $g''(0)$ when $S_w = 0$ are $c = -0{\cdot}3$ and $c = -0{\cdot}4294$, respectively.

Heat transfer relations

The characteristic quantity representing the heat transfer at the surface is $\mathrm{Nu}/\mathrm{Re}_w^{\frac{1}{2}}$, where the Nusselt number $\mathrm{Nu} = x_1(\partial T/\partial x_3)_w/(T_0 - T_w)$, and the Reynolds number $\mathrm{Re}_w = ax_1^2/\nu_w$. In terms of the dimensionless enthalpy gradient at the surface

$$\mathrm{Nu}/\mathrm{Re}_w^{\frac{1}{2}} = -S'(0)/S_w = -h'(0). \tag{4.4}$$

Values of $h'(0)$ are given in table 1 and graphically in figure 1. It is seen that $\mathrm{Nu}/\mathrm{Re}_w^{\frac{1}{2}}$ decreases with increasing c reaching a minimum near the point of onset of flow reversal where it now begins to increase. A possible explanation of this is that when flow reversal just occurs the x_2-velocity component is nearly zero in the neighbourhood of the surface such that heat is being transferred from the surface by conduction and by convection due to the x_1- and x_3-components of velocity. Since the x_1-component of velocity varies slowly with c it follows that $\mathrm{Nu}/\mathrm{Re}_w^{\frac{1}{2}}$ must increase when the flow reversal in the x_2-component of velocity becomes established.

Finally, from expression (4) the heat transfer coefficient

$$h_c = (k_w/x_1)\,\mathrm{Nu} = k_w(a/\nu_w)^{\frac{1}{2}}\,\mathrm{Nu}/\mathrm{Re}_w^{\frac{1}{2}}. \tag{4.5}$$

The local rate of heat transfer to the surface per unit surface area at the point of attachment, defined by the relation

$$q = h_c(T_0 - T_w),$$

is

$$q = k_w(T_0 - T_w)(a/\nu_w)^{\frac{1}{2}}\,\mathrm{Nu}/\mathrm{Re}_w^{\frac{1}{2}}. \tag{4.6}$$

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